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Characterization of the Zygmund Space by Shifted B-Splines

Peter Wingren

Department of Mathematics, University of Umeå, S-901 87 Umeå, Sweden
E-mail: Peter.Wingren@math.umu.se

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A method to decompose real valued continuous functions defined on **R** is put forward. The decomposition is by infinite linear combinations of B-splines. It is proved that a necessary and sufficient condition for a function to be in the Zygmund space is that the corresponding sequence of coefficients be in the sequence space ℓ^{∞}. © 2001 Academic Press

Key Words: Zygmund space; B-spline.

1. INTRODUCTION

Consider the class $\{\sum_{p=1}^{\infty} \sum_{i=-\infty}^{\infty} c_{p,i} 2^{1-p} B(2^{p-1}x - i) : \{\{c_{p,i}\}\} \in \ell^{\infty}\}$ where $B(x) = \operatorname{dist}(x, (-\infty, 0) \cup (2, \infty))$. It is well known that this class of functions contains other functions than the Zygmund functions. One example is the Takagi [1] nowhere differentiable function. We get this function from the given class by choosing $c_{p,2n} = 1$ and $c_{p,2n+1} = 0$ for all positive integers p and all integers n. This function has cusps so it can not be a function in the Zygmund class; as a matter of fact it has cusps on a set which is dense in **R**. In this paper it is proved that one gets the Zygmund space from the given class by shifting the B-splines slightly, or to be precise by replacing $B(2^{p-1}x - i)$ by $B(2^{p-1}x - i - \frac{1}{2})$.

There are many ways of characterizing the space of Zygmund functions e.g., by differences [2], polynomial approximation [2], trigonometric functions [3] and by wavelets [4]. The result in this paper is close to those in [5]. The main advantage of characterizing the Zygmund space, as we do here, by a frame [6] of B-splines is that it applies to fairly general subsets of **R** [7].

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2. DEFINITIONS, THEOREMS AND PROOFS

In the following we often exclude the independent variable in the notation for functions. We write for example f instead of $f(x)$. When we project $f(x)$ by spline operators (see the definitions of σ_p and s_p below) we often neglect to write the f, that is we write σ_p and s_p for $\sigma_p(f)$ and $s_p(f)$ respectively.

The paragraphs which follow contain definitions and some preparations which pave the way for the main results, Theorems 1 and 2.

DEFINITION 1. The real-valued function $f(x)$ belongs to the Zygmund space $\Lambda(1, \mathbf{R})$ if there is a constant $M > 0$ so that

$$|f(x)| \leqslant M, \qquad x \in \mathbf{R} \tag{1}$$

and

$$|\Delta_h^2 f(x)| \leqslant Mh, \qquad x \in \mathbf{R}, \qquad h \in [0, 1]. \tag{2}$$

The infimum of all constants M above defines a norm, $\|\cdot\|_\Lambda$, on $\Lambda(1, \mathbf{R})$. It is well known that the pair $(\Lambda(1, \mathbf{R}), \|\cdot\|_\Lambda)$ is a Banach space and that the functions in the Zygmund space are continuous functions. We have as a matter of fact for small values of h that

$$f(x) \in \Lambda(1, \mathbf{R}) \Rightarrow \text{there is a } C > 0 \text{ so that } |\Delta_h f(x)| \leqslant Ch\,|\log h|.$$

The following result is also well known and will be useful for us. For every first degree polynomial $P_{a, a+h}(x)$ interpolating $f(x)$ at a and $a+h$ we have

$$|f(x) - P_{a, a+h}(x)| \leqslant \frac{\|f\|_\Lambda}{3} h \qquad \text{where} \quad x \in [a, a+h]. \tag{3}$$

In relation to knot point sets we use the following notation. If A is a countably infinite and unbounded (from below and above) subset of the real numbers and if $\{a_i\}_{i=-\infty}^{\infty}$, $\cdots a_{i-1} < a_i < a_{i+1} < \cdots$ is the corresponding sequence then we let $I(A) := \{(a_i, a_{i+1}], i \in \mathbf{Z}\}$.

We will now specify a sequence of knot point sets and two corresponding sequences of spline functions. Let $K_0 = \mathbf{Z}$, K_1 be the set of all midpoints of intervals in $I(K_0)$, K_2 be the set of all midpoints of intervals in $I(K_0 \cup K_1)$, K_3 be the set of all midpoints of intervals in $I(K_0 \cup K_1 \cup K_2)$ and so on. For convenience we also call K_0 a knot point set. From the above construction it can be seen that the knot point sets $K_0, K_1, K_2, \ldots$ have the following properties.

1. $K_p = 2^{1-p}(\mathbf{Z} + \frac{1}{2})$, $p = 1, 2, ...$,
2. $\operatorname{dist}(K_i, K_p) = 2^{-p}$, $i < p$ and $p = 1, 2, ...$,
3. every interval of $I(K_p)$ contains exactly one point from $K_0 \cup \cdots \cup K_{p-1}$ $p = 1, 2, ...$ and
4. every second interval from $I(K_p)$ contains exactly one point from K_{p-1} and every other second interval contains no point from K_{p-1}. Hence every second interval of $I(K_p)$ is a subset of an interval in $I(K_{p-1})$ and every other second is not.

Given a real-valued function $f(x)$, $x \in \mathbf{R}$, we make the following definitions.

Definition 2. Let $\sigma_p[f]$ be the linear spline which interpolates f and has knot points at K_p, $p = 1, 2, ...$

Definition 3. Let $s_0[f] = 0$ and $s_p[f]$ be the linear spline which interpolates $f - (s_0 + s_1 + \cdots + s_{p-1})[f]$ and has knot points at K_p, $p = 1, 2, ...$, that is

$$s_p[f] := \sigma_p[f - (s_0 + s_1 + \cdots + s_{p-1})[f]].$$

Our goal is now to prove that $(s_1 + s_2 + \cdots + s_p)[f]$, $p = 1, 2, ...$, converge to f as fast as $\sigma_p[f]$ do. From (3) we know that

$$|f - \sigma_p[f]| \leqslant \frac{\|f\|_\Lambda}{3} 2^{1-p}$$

so we would like to prove that

$$|(s_1 + s_2 + \cdots + s_p)[f] - f| \leqslant c \frac{\|f\|_\Lambda}{3} 2^{1-p}$$

for some constant c. Since

$$\begin{aligned} &|(s_1 + s_2 + \cdots + s_p)[f] - f| \\ &\quad \leqslant |(s_1 + s_2 + \cdots + s_p)[f] - \sigma_p[f]| + |\sigma_p[f] - f|, \end{aligned}$$

we will reach our goal if we prove that

$$|(s_1 + s_2 + \cdots + s_p)[f] - \sigma_p[f]| \leqslant c_o \frac{\|f\|_\Lambda}{3} 2^{1-p}$$

for some constant c_o. This will be done in Theorem 1. But first we state and prove a lemma which will be useful in the theorem.

LEMMA 1. *Let a sequence of sequences be given in the following way. Sequence number* 1 *has terms whose absolute values are less than or equal to* 1. *If sequence number* p, $p \geqslant 1$, *is*

$$(..., a, b, ...)$$

then sequence number $p+1$ *is*

$$\left(..., a, -\frac{a+b}{2}+c, b, ...\right),$$

where $|c| \leqslant 1$; *thus, sequence number* $p+1$ *is generated from number* p *by inserting between every pair of elements, say* a *and* b, *a new element* $-\frac{a+b}{2}+c$, *where* $c = c(a, b)$ *is an arbitrary number in* $[-1, 1]$. *Then all the given sequences are bounded by* 3.

Proof. We prove by induction. Let $S(n)$ be the statement that the elements in sequence number n are bounded by 3. It is obvious that $S(n)$ is true for $n \leqslant 2$. Now assume that $S(n)$ is true for $n \leqslant p$ and let sequence number $p-1$ be

$$(..., a, b, ...).$$

Then sequence number $p+1$ is

$$\left(..., a, -\frac{1}{4}a+\frac{1}{4}b-\frac{1}{2}c+d, -\frac{a+b}{2}+c, \frac{1}{4}a-\frac{1}{4}b-\frac{1}{2}c+e, b, ...\right),$$

where c, d and e are arbitrary numbers in $[-1, 1]$. The new elements that are added in sequence $p+1$ may then be estimated like

$$\left|-\tfrac{1}{4}a+\tfrac{1}{4}b-\tfrac{1}{2}c+d\right| \leqslant \tfrac{1}{4}|a|+\tfrac{1}{4}|b|+\tfrac{1}{2}|c|+|d| \leqslant \tfrac{1}{4}3+\tfrac{1}{4}3+\tfrac{1}{2}+1=3.$$

Hence by induction we get that $S(n)$ is true for all positive integers and the proof of the lemma is complete.

We now introduce the notation

$$\lambda_p[f] := (s_1+s_2+\cdots+s_p)[f]-\sigma_p[f] \tag{4}$$

and prove the following theorem.

THEOREM 1. *Let $f \in \Lambda(1, \mathbf{R})$. Then*

$$\lambda_{p+1}[f] = \lambda_p[f] - \sigma_{p+1}[\lambda_p[f]] + \sigma_p[f] - \sigma_{p+1}[\sigma_p[f]] \tag{5}$$

and

$$|\lambda_p[f]| \leqslant \tfrac{3}{2} \, \|f\|_\Lambda \, 2^{-p}. \tag{6}$$

Proof. Let $f \in \Lambda(1, \mathbf{R})$. We divide the proof into three parts labeled (I), (II) and (III). In part (I) we derive the recursion formula (5), in part (II) we estimate $\sigma_p[f] - \sigma_{p+1}[\sigma_p[f]]$ and in part (III) we estimate $\lambda_p[f] - \sigma_{p+1}[\lambda_p[f]]$ and combine the estimates to get (6). Since all the operators and compositions of operators in this proof work on f we will usually neglect to write out the f in the equations and inequalities which follow.

(I) It is obvious from Definition 1 and Definition 2 that

$$s_1 + s_2 + \cdots + s_{p+1} = \sigma_{p+1} \qquad \text{on} \quad K_{p+1}.$$

Then

$$s_{p+1} - \sigma_{p+1} = \sigma_p - (s_1 + s_2 + \cdots + s_p) - \sigma_p \qquad \text{on} \quad K_{p+1}$$

and by (4)

$$s_{p+1} - \sigma_{p+1} = -\lambda_p - \sigma_p \qquad \text{on} \quad K_{p+1}.$$

We apply σ_{p+1} to this equation. Since σ_{p+1} is a projection on linear splines with knots on K_{p+1} we get

$$s_{p+1} - \sigma_{p+1} = -\sigma_{p+1}(\lambda_p) - \sigma_{p+1}(\sigma_p)$$

and by adding $s_1 + \cdots + s_p$ to both sides and $-\sigma_p + \sigma_p$ to the right hand side we get

$$s_1 + \cdots + s_p + s_{p+1} - \sigma_{p+1} = s_1 + \cdots + s_p - \sigma_p - \sigma_{p+1}(\lambda_p) + \sigma_p - \sigma_{p+1}(\sigma_p).$$

Now recognizing in this equation the expressions equal to λ_{p+1} and λ_p respectively and making the corresponding substitution, we get (5).

(II) By property (4) for knot points we know that there are no point from K_p in every second interval of $I(K_{p+1})$. It follows that the restriction of σ_p to such an interval is a linear function. Hence σ_p and $\sigma_{p+1}(\sigma_p)$ are equal and thus $\sigma_p - \sigma_{p+1}(\sigma_p)$ vanishes on those intervals. But on every other second interval of $I(K_{p+1})$ there is exactly one point from K_p and it is obvious that $\sigma_p - \sigma_{p+1}(\sigma_p)$ restricted to such an interval has a local extreme value at the K_p point. Since $2^{1-p}(i+\frac{1}{2})$ is a point in K_p we have by Definition 2 that

$$\sigma_p(2^{1-p}(i+\tfrac{1}{2})) = f(2^{1-p}(i+\tfrac{1}{2}))$$

and by (2) that

$$\left|\sigma_p\left(2^{1-p}\left(i+\frac{1}{2}\right)\right) - \sigma_{p+1}(\sigma_p)\left(2^{1-p}\left(i+\frac{1}{2}\right)\right)\right|$$

$$= \frac{1}{4}\left|f\left(2^{1-p}\left(i+\frac{1}{2}\right)\right) - \frac{f(2^{1-p}(i+\frac{1}{2}-1) + f(2^{1-p}(i+\frac{1}{2}+1)}{2}\right|$$

$$\leqslant \frac{\|f\|_\Lambda}{4} 2^{-p}.$$

The last inequality follows from (2) by breaking out a 2 and replacing h by $h/2$ and then substituting $x = 2^{1-p}(i+\frac{1}{2}-1)$ and $h = 2^{2-p}$.

(III) We now prove (6) and this is done by means of Lemma 1. But to be able to apply the lemma we need some preparation.

From (5) it follows easily that (6) holds for $p=1$ and $p=2$. For $p>2$, we argue in the following way. The function λ_{p+1} is a linear spline with knots in $K_1 \cup K_2 \cup \cdots \cup K_{p+1}$. Hence we just have to check the values at those knot points. But $\lambda_{p+1} = 0$ at points in K_{p+1} and hence just the values at $K_1 \cup K_2 \cup \cdots \cup K_p$ remain to be checked. We do this by investigating the right hand side of Eq. (5).

In the evaluations and estimations below we rely on the distribution of the knot points and the piecewise linearity of the λ_p-functions. We have for instance

$$\lambda_p(x) = \lambda_p(\tfrac{1}{2}(x-d) + \tfrac{1}{2}(x+d)) = \tfrac{1}{2}\lambda_p(x-d) + \tfrac{1}{2}\lambda_p(x+d)$$

if λ_p is linear on $[x-d, x+d]$. Assume now that $x \in K_0 \cup K_1 \cup \cdots \cup K_p$, $d = 2^{-p}$ and investigate the difference of the first two terms in (5). Since $\sigma_{p+1}(\lambda_p)$ is linear on $[x-d/2, x+d/2]$ and furthermore interpolates λ_p at $x+d/2$ and $x-d/2$ and λ_p is linear on both $[x-d, x]$ and $[x, x+d]$ we get

$$\begin{aligned}
\sigma_{p+1} \circ \lambda_p(x) &= \sigma_{p+1} \circ \lambda_p\left(\frac{1}{2}(x-d) + \frac{1}{2}(x+d)\right) \\
&= \frac{1}{2}\left(\sigma_{p+1} \circ \lambda_p\left(x - \frac{d}{2}\right) + \sigma_{p+1} \circ \lambda_p\left(x + \frac{d}{2}\right)\right) \\
&= \frac{1}{2}\left(\lambda_p\left(x - \frac{d}{2}\right) + \lambda_p\left(x + \frac{d}{2}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\left(\lambda_p\left((x - \frac{d}{2}\right) - \frac{d}{2}\right) + \lambda_p\left(\left(x - \frac{d}{2}\right) + \frac{d}{2}\right)\right)\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{2}\left(\lambda_p\left(x + \frac{d}{2}\right) - \frac{d}{2}\right) + \lambda_p\left(\left(x + \frac{d}{2}\right) + \frac{d}{2}\right)\right)\right) \\
&= \frac{1}{2}\left(\lambda_p(x) + \frac{1}{2}\lambda_p(x-d) + \frac{1}{2}\lambda_p(x+d)\right). \qquad (7)
\end{aligned}$$

We now evaluate the last expression for two cases, first $x \in K_0 \cup K_1 \cup \cdots \cup K_{p-1}$ and then $x \in K_p$. If we have the first case, then two terms in the last expression vanish,

$$\lambda_p(x-d) = 0 \qquad \text{and} \qquad \lambda_p(x+d) = 0$$

and it follows that

$$\sigma_{p+1} \circ \lambda_p(x) = \tfrac{1}{2}\lambda_p(x).$$

Since $x \in K_0 \cup K_1 \cup \cdots \cup K_{p-1}$ we get for the difference of the two last terms in (5) that

$$(\sigma_p - \sigma_{p+1} \circ \sigma_p)(x) = 0$$

and if we use the last two formulas we may simplify formula (5) to

$$\lambda_{p+1}(x) = \tfrac{1}{2}\lambda_p(x), \qquad x \in K_0 \cup K_1 \cup \cdots \cup K_{p-1}. \qquad (8)$$

The second case is $x \in K_p$. Then, since $\lambda_p(x) = 0$ it follows from (7) that

$$\sigma_{p+1} \circ \lambda_p(x) = \tfrac{1}{2}(\tfrac{1}{2}\lambda_p(x-d) + \tfrac{1}{2}\lambda_p(x+d))$$

and hence if we insert this in (5) we get

$$\lambda_{p+1}(x) = -\tfrac{1}{2}(\tfrac{1}{2}\lambda_p(x-d) + \tfrac{1}{2}\lambda_p(x+d)) + (\sigma_p - \sigma_{p+1}(\sigma_p))(x), \qquad (9)$$

where $x-d$ and $x+d$ are the two closest neighbours of x in $K_0 \cup K_1 \cup \cdots \cup K_{p-1}$.

To make it possible to apply Lemma 1 we now introduce the following substitutions. Let

$$\lambda_{p+1}(x) = 2^{-(p+2)} \|f\|_{\Lambda} \mu_{p+1}(x)$$

and

$$(\sigma_p - \sigma_{p+1}(\sigma_p))(x) = 2^{-(p+2)} \|f\|_{\Lambda} \tau_{p+1}(x), \qquad \text{for} \quad p = 0, 1, 2, ...$$

Then formulas (8) and (9) transform into

$$\mu_{p+1}(x) = \mu_p(x), \qquad x \in K_0 \cup K_1 \cup \cdots \cup K_{p-1}$$

and

$$\mu_{p+1}(x) = -\tfrac{1}{2}(\mu_p(x-d) + \mu_p(x+d)) + \tau_{p+1}(x), \qquad \text{where} \quad x \in K_p.$$

It is now easy to see from (5) that by the substitution we just made we have

$$\mu_1 = 0 \qquad \text{and} \qquad |\mu_2| \leqslant 1.$$

We see moreover from (8) and (9) that the sequence

$$\mu_p(x), \qquad x \in K_0 \cup K_1 \cup \cdots \cup K_{p-1}$$

generates the sequence

$$\mu_{p+1}(x), \qquad x \in K_0 \cup K_1 \cup \cdots \cup K_p$$

in such a way that it satisfies the assumptions in Lemma 1. It follows that

$$|\mu_{p+1}(x)| \leqslant 3.$$

Hence by the substitution formulas we get

$$|\lambda_{p+1}(x)| \leqslant \tfrac{3}{4} \|f\|_{\Lambda} 2^{-p},$$

thus

$$|\lambda_p(x)| \leqslant \tfrac{3}{2} \|f\|_{\Lambda} 2^{-p}$$

and by that the proof is complete.

For every real valued continuous function $f(x)$, x real, we have by Definition 3 a unique sequence of spline functions $s_p(x)$ and there is a unique way of writing $s_p(x)$ by the B-spline $B(x)$, which was defined in the Introduction,

$$s_p(x) = \sum_{i=-\infty}^{+\infty} c_{p,i}(f)\, 2^{1-p} B(2^{p-1}x - i - \tfrac{1}{2}).$$

This means that we get a transformation $T(f)$ which maps functions to sequences

$$T(f) = \{\{c_{p,i}(f)\}_{i=-\infty}^{\infty}\}_{p=1}^{\infty}.$$

We also define a linear transformation S on ℓ^∞ by

$$S(\{\{c_{p,i}\}_{i=-\infty}^{\infty}\}_{p=1}^{\infty}) = \sum_{p=1}^{\infty} \sum_{i=-\infty}^{+\infty} c_{p,i} 2^{1-p} B(2^{p-1}x - i - \tfrac{1}{2}).$$

We now state and prove our main theorem.

THEOREM 2. *Given a function $f: \mathbf{R} \to \mathbf{R}$. Then $f \in \Lambda(1, R)$ if and only if $T(f) \in \ell^\infty$. Moreover there are $c_1, c_2 > 0$ so that*

$$\|T(f)\|_\infty \leqslant c_1 \|f\|_\Lambda \qquad \text{for all} \quad f \in \Lambda \tag{10}$$

and if $\{\{c_{p,i}\}_{i=-\infty}^{\infty}\}_{p=1}^{\infty} \in \ell_\infty$ then

$$\|S(\{\{c_{p,i}\}_{i=-\infty}^{\infty}\}_{p=1}^{\infty})\|_\Lambda \leqslant c_2 \|\{\{c_{p,i}\}_{i=-\infty}^{\infty}\}_{p=}^{\infty}\|_\infty. \tag{11}$$

Proof. In the proof we write $\{\{c_{p,i}\}\}$ for $\{\{c_{p,i}\}_{i=-\infty}^{\infty}\}_{p=1}^{\infty}\}$. It is obvious that the theorem is proved if we prove the inequalities (10) and (11). To prove (10) let $f \in \Lambda(1, \mathbf{R})$ and let s_p and σ_p, $p = 1, 2, ...$ be the functions given by Definitions 2 and 3. Then by (3) we have

$$|f(x) - \sigma_p(x)| \leqslant \frac{\|f\|_\Lambda}{3} 2^{1-p}$$

and from (6) it now follows that

$$\begin{aligned} |s_p| &= |s_p + s_{p-1} + \cdots + s_1 - \sigma_p + \sigma_p - \sigma_{p-1} + \sigma_{p-1} \\ &\quad - (s_{p-1} + s_{p-2} + \cdots + s_1)| \leqslant |s_p + s_{p-1} + \cdots + s_1 - \sigma_p| \\ &\quad + |\sigma_p - \sigma_{p-1}| + |\sigma_{p-1} - (s_{p-1} + s_{p-2} + \cdots + s_1)| \\ &\leqslant \tfrac{3}{2} \|f\|_\Lambda 2^{-p} + 2 \|f\|_\Lambda 2^{-p} + 3 \|f\|_\Lambda 2^{-p} = \tfrac{31}{6} \|f\|_\Lambda 2^{-p} \end{aligned}$$

and the inequality (10) is proved with $c_1 = \frac{31}{6}$.

To show the second inequality in Theorem 2 let $\{\{c_{p,i}\}\} \in \ell^\infty$ be given and let us prove that the $\Lambda(1, \mathbf{R})$-norm of $S(\{\{c_{p,i}\}\})(x)$ satisfies the inequality (11). We begin by establishing the first property of the norm in Definition 1, that is, that $S(\{\{c_{p,i}\}_{i=-\infty}^{\infty}\}_{p=1}^{\infty})$ is bounded. Indeed,

$$
\begin{aligned}
|S(\{\{c_{p,i}\}\})(x)| &\leqslant \sum_{p=1}^{\infty} 2^{1-p} \left| \sum_{i=-\infty}^{\infty} c_{p,i} B(2^{p-1}x - i - \tfrac{1}{2}) \right| \\
&\leqslant \sum_{p=1}^{\infty} 2^{1-p} \, \|\{\{c_{p,i}\}\}\| \leqslant 2 \, \|\{\{c_{p,i}\}\}\|.
\end{aligned}
$$

We now prove inequality (11) for the sequence

$$
\{\{a_{p,i}\}\} = \left\{\left\{ \frac{c_{p,i}}{\|\{\{c_{p,i}\}\}\|_\infty} \right\}\right\}.
$$

It then follows from linearity that the inequality holds for $\{\{c_{p,i}\}\}$. Given $h \in (0, 1]$ choose a positive integer N so that

$$
2^{-1-N} \leqslant |2h| < 2^{-N}.
$$

Then the interval $[x, x+2h]$ contains at most one point from $\bigcup_{i=0}^{N} K_i$ and hence all except three terms in $\sum_{p=1}^{\infty} \sum_{i=-\infty}^{\infty} 2^{1-p} c_{p,i} B(2^{p-1}x - i - \frac{1}{2})$ are first degree polynomials on the interval. This means that the second differences of $B(2^{p-1}x - i - \frac{1}{2})$, $p \leqslant N$, $i \in \mathbf{Z}$ vanish, except for at most three of them, and the absolute values of these are at most h, $2h$ and h. It follows that

$$
\begin{aligned}
&|\Delta_h^2 S(\{\{a_{p,i}\}\})(x)| \\
&\quad = \left| \Delta_h^2 \sum_{p=1}^{\infty} \sum_{i=-\infty}^{\infty} a_{p,i} 2^{1-p} B(2^{p-1}x - i - \tfrac{1}{2}) \right| \\
&\quad \leqslant \left| \Delta_h^2 \sum_{p=1}^{N} \sum_{i=-\infty}^{\infty} a_{p,i} 2^{1-p} B(2^{p-1}x - i - \tfrac{1}{2}) \right| \\
&\quad\quad + \left| \Delta_h^2 \sum_{p=N+1}^{\infty} \sum_{i=-\infty}^{\infty} a_{p,i} 2^{1-p} B(2^{p-1}x - i - \tfrac{1}{2}) \right| \\
&\quad \leqslant 4h \, \|\{\{a_{p,i}\}\}\| + \left| \Delta_h^2 \sum_{p=N+1}^{\infty} \sum_{i=-\infty}^{\infty} a_{p,i} 2^{1-p} B(2^{p-1}x - i - \tfrac{1}{2}) \right|.
\end{aligned} \tag{12}
$$

Since $\|\{\{a_{p,i}\}\}\| = 1$ and since the double sum in the last inequality is a function bounded by $2^{-(N-1)}$ we get

$$|\Delta_h^2 S(\{\{a_{p,i}\}\})(x)| \leqslant 4h + 4 \cdot 2^{-(N-1)}.$$

If we then use (2) we get

$$|\Delta_h^2 S(\{\{a_{p,i}\}\})(x)| \leqslant 4h + 32h = 36h.$$

Thus the inequality (11) is proved with $c_2 = 36$ and by that the proof is complete.

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